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WZW models of general simple groups

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Abstract

It is shown that a WZW model corresponding to a general simple group possesses in general different quantisations which are parametrised by $\mathrm{Hom}(\pi_1(G), \mathrm{Hom}(\pi_1(G), U(1)))$. The quantum theories are generically neither monodromy nor modular invariant, but all the modular invariant theories of Felder *et.al.* are contained among them.

A formula for the transformation of the Sugawara expression for L_0 under conjugation with respect to non-contractible loops in LG is derived. This formula is then used to analyse the monodromy properties of the various quantisations. It turns out that for $\pi_1(G) \cong \mathbb{Z}_N$, with N even, there are 2 monodromy invariant theories, one of which is modular invariant, and for $\pi_1(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ there are 8 monodromy invariant theories, two of which are modular invariant. A few specific examples are worked out in detail to illustrate the results.

1 Introduction

Among the various conformal field theories, the Wess-Zumino-Witten (WZW) models [30, 24, 15] take a somewhat special position. First of all, due to the vast knowledge about the representation theory of the affine Kac-Moody algebras [28, 22, 16], their mathematical

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structure is well understood from an algebraic point of view. On the other hand the models possess a formulation in terms of an action, and thus more conventional techniques may be used. In addition, even though rather special, a very large class of conformal field theories can be constructed from WZW models, using the coset construction [18].

Most of the work which has been done on WZW models has only taken into account the local structure of the underlying (target space) group, ignoring global topological effects. In this paper we shall try to understand some of these global issues. In particular we shall be interested in WZW models of groups which are not simply-connected.

Whereas the algebraic approach to WZW models is more powerful for local considerations, the formulation in terms of an action allows a discussion of the global issues, and we shall thus take it as our starting point. We shall explain how the theory can be quantised, and show how the spectrum of the corresponding quantum theories can be described algebraically. We shall then analyse how many inequivalent quantisations exist and exhibit them explicitly. We find that the various quantisations are parametrised by $\mathrm{Hom}(\pi_1(G), \mathrm{Hom}(\pi_1(G), U(1)))$, where $\pi_1(G)$ is the fundamental group of the group G under consideration.

Having given the various quantisations explicitly, we shall study some of their properties in detail. In particular, we shall analyse the behaviour of the fields under monodromy, *i.e.* the analytic continuation of one field about another one, and we shall show that, for general quantisations, the operator product transforms with respect to a (non-trivial) one-dimensional representation. This implies in particular that the amplitudes are only defined on some covering space. The appearance of an 'anyonic' representation is quite generic for two-dimensional quantum field theories (see *e.g.* [12, 9]), and it suggests that the theory is genuinely braided. In a string theory inspired context, such theories have traditionally been excluded; however, from the point of view of Euclidean conformal field theory, this restriction does not seem to be justified.

The class of theories for which all operator products are invariant under monodromy, the *monodromy invariant theories*, are of special interest, not least from the traditional point of view of Euclidean conformal field theory. We shall show that, depending on the structure of the fundamental group of G, one, two or eight of the different quantisations are monodromy invariant. In more detail, for $\pi_1(G) \cong \mathbb{Z}_N$, there exists one monodromy invariant theory, unless N is even, in which case there are two. One of the monodromy invariant theories is the known modular invariant theory [3, 8], and the other (for even N) is not modular invariant. For $\pi_1(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, there are eight monodromy invariant quantisations, two of which are the modular invariant theories of Felder *et.al.* [8], and the other six are not modular invariant.

The essential calculational tool for the analysis of the monodromy is a formula for the adjoint action of non-contractible loops in LG on the generator L_0 which we derive. This formula may have some interest in its own right as the action of the non-contractible loops in LG on $\mathbb{R} \oplus l\mathfrak{g} \oplus \mathbb{R}$, the Lie algebra of $\Pi \triangleright \tilde{L}\widetilde{G}$ [28], a priori involves a choice; here this choice is fixed by identifying the generator of the rigid rotations, L_0, with the Sugawara expression which is quadratic in the Lie algebra of $\tilde{L}\widetilde{G}$, $\hat{\mathfrak{g}} \cong l\mathfrak{g} \oplus \mathbb{R}$.

The paper is organised as follows. We start in section 2 by studying the quantisation of the theory defined by an action, and explain how the different quantisations arise, giving explicit formulae for the spectrum of all possible quantisations. In section 3, we determine the adjoint action of non-contractible loops in LG on the generator L_0, and show that all quantisations transform with respect to a one-dimensional representation under monodromy. We then analyse in section 4 which quantisations are monodromy invariant. In section 5, some examples for the additional (monodromy invariant) theories are exhibited in detail, and section 6 contains a few concluding remarks. In the appendix, we recall some of the less widely known facts about the affine Weyl group, and give a new geometrical proof for an old observation of Olive and Turok [26] about the symmetries of the affine Dynkin diagram.

2 The different quantisations

We want to consider the theory defined by the WZW action [30, 7, 8]

$$\mathcal{S}[g] = -\frac{k}{4\pi}\int_{\mathcal{M}} \left\langle g^{-1}\partial g, g^{-1}\bar{\partial} g\right\rangle - \frac{k}{24\pi}\int_{\mathcal{B}} \left\langle \tilde{g}^{-1}d\tilde{g}, [\tilde{g}^{-1}d\tilde{g}, \tilde{g}^{-1}d\tilde{g}]\right\rangle \, . \tag{2.1}$$

Here the field $g:\mathcal{M}\rightarrow G$ takes values in a simple, connected, compact group G, and $\tilde{g}$ is an extension of g to $\mathcal{B}$ where $\partial\mathcal{B}=\mathcal{M}$. $\mathcal{M}$ is the two-dimensional space-time, and we take space to be compactified so that $\mathcal{M}=S^1\times \mathbb{R}$. $\langle .,.\rangle$ is the Killing form on the Lie algebra $\mathfrak{g}$ of G, normalised so that the longest roots of the algebra have length square equal to 2.
We shall mainly be interested in the case where G is not simply-connected. G can then be written as

$$G = \widetilde{G}/C \, , \tag{2.2}$$

where $\widetilde{G}$ is the universal covering group of G, and C is a subgroup of the centre $\mathcal{Z}$ of $\widetilde{G}$. The centre is isomorphic to $\mathbb{Z}_N$ (for some N) for all simply-connected simple compact groups, with the exception of $D_{2n}\cong SO(4n)$ for which the centre is $\mathbb{Z}_2\times\mathbb{Z}_2$.
The second term in (2.1), the Wess-Zumino term, depends on the choice of the extension $\tilde{g}$, and the consistency of the quantum theory requires this ambiguity to be $2\pi\mathbb{Z}$. This imposes quantisation conditions on k. For $C=\mathbb{Z}_N$, the quantisation condition is [8]

$$\frac{kN}{2}\langle c_1, c_2\rangle \in \mathbb{Z}\, , \tag{2.3}$$

for all $c_i\in C$, and for $C=\mathbb{Z}_2\times\mathbb{Z}_2$, the condition is

$$k\langle c_1,c_1\rangle, \quad k\langle c_2,c_2\rangle, \quad k\langle c_1,c_2\rangle \in \mathbb{Z}\, , \tag{2.4}$$

where $c_1=(1,0)$ and $c_2=(0,1)$. Here we have identified (as we shall do from now on) $\hat{c}\in C\subset\mathcal{Z}\subset\widetilde{G}$ with c in the quotient space of a Cartan subalgebra $\mathfrak{h}$ by the coroot lattice via $\hat{c}=exp(c)$. For $C=\mathbb{Z}_2\times\mathbb{Z}_2$, the definition of the Wess-Zumino term remains ambiguous and there are at least two quantisations [8].

Recall that the most general solution $g(x,t)$ of the (classical) WZW-model is of the form

$$g(x,t) = u(x^+) \cdot v(x^-)\,, \tag{2.5}$$

where $x^\pm = t \pm x$ and $u, v : \mathbb{R} \to G$. The solution g has to be periodic in $x \mapsto x + 2\pi$, and this implies that u and v have to satisfy

$$u(z+2\pi) = u(z)M, \qquad v(z+2\pi) = Mv(z)\,, \tag{2.6}$$

where M is some group element. The pair of functions (u,v) does not uniquely determine the solution g; the transformation

$$u \longrightarrow ug_0 \qquad v \longrightarrow g_0^{-1}v\,, \tag{2.7}$$

under which

$$M \longrightarrow g_0^{-1}Mg_0 \tag{2.8}$$

leaves g invariant. We can use this freedom to rotate M into a fixed maximal torus H of G. This fixes the pair of functions (u,v) generically up to

$$u \longrightarrow un \qquad v \longrightarrow n^{-1}v\,, \tag{2.9}$$

where $n \in N(H)$, the normaliser of H, under which $M \longrightarrow n^{-1}Mn$ [5]. (If M is not regular, *i.e.* if M belongs to more than one maximal torus, the residual symmetry is even larger.) The action of n on M has kernel H; the quotient $\mathcal{W} = N(H)/H$ is the Weyl group of G.

The configuration space of the WZW model is the space of functions $\mathcal{M} \to G$. We could now attempt to describe the phase space of the system in terms of coordinates (on this configuration space) and their conjugate momenta, but this would seem to be rather difficult. We shall therefore use a different description which was already employed in [5] and goes back to [6]. In this approach, the phase space is regarded as the manifold of solutions of the equations of motion of the field theory. As in [5], this space can be regarded as a symplectic quotient of a larger space in which we relax the constraint that the monodromy of u and v should be the same. That is, we introduce $\nu_L, \nu_R \in \mathfrak{h}$, the Lie algebra of the maximal torus H, and write the left and right monodromy, respectively, as

$$M_L = exp(2\pi\nu_L) \qquad M_R = exp(2\pi\nu_R)\,. \tag{2.10}$$

We can then move all the non-trivial homotopic information about u and v into ν_L and ν_R, respectively, *i.e.* we can write

$$u(x^+) = \tilde{u}(x^+)exp(\nu_L x^+) \tag{2.11}$$

and

$$v(x^-) = exp(\nu_R x^-)\tilde{v}(x^-)\,, \tag{2.12}$$

where $\tilde{u}, \tilde{v} \in L\widetilde{G}$, the loop group of the simply connected covering group $\widetilde{G}$ of G. We can restrict ν_L and ν_R, without loss of generality, to lie in a fixed alcove (a chamber of the Stiefel diagram) of $\mathfrak{g}$ [4]. Then the phase space is the quotient of the submanifold $exp(2\pi\nu_L) = exp(2\pi\nu_R)$ (as a relation in G) by the action of $(H \times C)$,

$$\cup_{c\in C} \left((\tilde{u}, \nu_L, \nu_R, \tilde{v})|_{\nu_L = c\cdot\nu_R} \right) / (H \times C) \, , \tag{2.13}$$

where H acts on $\tilde{u}$ and $\tilde{v}$ as in (2.9) and leaves ν_L and ν_R invariant, and $c \in C$ maps $(\tilde{u}, \nu_L, \nu_R, \tilde{v})$ to $(\tilde{u}, \nu_L, \nu_R, \tilde{v}\, c)$. To describe the action of $c \in C$ on ν_R we regard (as before) $C \cong \pi_1(G)$ as the quotient space of the lattice of integral elements of $\mathfrak{h}$, *i.e.* those that are mapped to 1 in G, by the coroot lattice; this quotient space acts naturally on a fixed alcove by translation.

To quantise the theory we should now find a subalgebra of functions on phase space which can be consistently defined as operators in the quantum theory, replacing Poisson brackets by commutators. Unfortunately, the phase space is rather complicated (in particular it is not a vector space), and it is therefore very difficult to find such a subalgebra explicitly. On the other hand, we might argue that the subalgebra should contain the analogues of the position and momentum function, and that therefore the phase space itself (regarded as a subspace of the space of distributions on phase space) should be contained in the closure of this subalgebra. Then, the quantum states should form a representation of (2.13), and thus of

$$\left(\cup_{c\in C} \, (\tilde{u}, \nu_L, \nu_R, \tilde{v})|_{\nu_L = c\cdot\nu_R} \right) \, , \tag{2.14}$$

which is covariant under the action of $(H \times C)$. (This description resembles strongly the formulation in [14].) For $G = \widetilde{G}$, C is the trivial group, and (2.13) consists of one component only. In this case the configuration space (and the phase space) is simply connected, and there should exist only one quantisation in which the quantum states form a representation of (2.14) which is invariant under H. This should force the representations of $\tilde{u}$ and $\tilde{v}$ to be conjugate, and we expect therefore that the spectrum only contains states in the diagonal theory. Indeed, it is known [8] that the theory for $\widetilde{G}$ is

$$\mathcal{H}_{\widetilde{G}} = \sum_j \mathcal{H}_j \otimes \mathcal{H}_{\bar{j}} \, , \tag{2.15}$$

where the sum extends over all unitary positive energy representations of $\tilde{L}\widetilde{G}$, the central extension of the loop group $L\widetilde{G}$ [28], and $\mathcal{H}_{\bar{j}}$ denotes the conjugate representation to $\mathcal{H}_j$.

In the general case, the spectrum of the theory corresponding to $G = \widetilde{G}/C$ contains also states which are representations of the other components of (2.14). The additional states of the quantum theory should be obtained by the action of the non-trivial loops of G (which are labelled by $c \in C$) on one of the two representations in the tensor products of (2.15). Recall from Pressley and Segal [28], that the action (by conjugation) of the non-trivial loops of G on $\tilde{L}\widetilde{G}$ is well-defined. This action does not preserve the set of positive roots, but, as has been

shown in [8] (see also the appendix), there exists a unique element in the affine Weyl group $\overline{\mathcal{W}}(\mathfrak{g})$, so that the composition with this affine Weyl group element preserves the positive roots. The non-trivial loops of G therefore induce (outer) automorphisms of $\tilde{L}\widetilde{G}$, and thus map in general a representations of $\tilde{L}\widetilde{G}$, $\mathcal{H}_j$, into a different representation which we denote by $\mathcal{H}_{c(j)}$. (The map $c(j)$ has been calculated for all simple groups in [8].)

Each component of the configuration space (and also of the phase space (2.13)) is not simply connected, the fundamental group being isomorphic to C. We therefore expect that there should be different quantisations of the classical theory corresponding to different monodromies with respect to C. [2] To classify the different quantisations, recall that for any subgroup C of the centre of $\widetilde{G}$, the irreducible representations of $\widetilde{G}$ (and thus also of $\tilde{L}\widetilde{G}$) fall into equivalence classes which are characterised by the induced (one-dimensional) representation of C. The tensor product of two irreducible representations is in the equivalence class corresponding to the product representation. For each sector, labelled by $c \in C$, let R_c denote the equivalence class of representations of $\tilde{L}\widetilde{G}$ which corresponds to the representation $R_c : C \to U(1)$. The possible quantisations are then of the form

$$\mathcal{H}_G^R := \sum_{c\in C} \sum_{j\in R_c} \mathcal{H}_j \otimes \mathcal{H}_{c(\bar{j})} \,, \tag{2.16}$$

where R_c has to satisfy

$$R_{c_1} R_{c_2} = R_{c_1+c_2} \,, \tag{2.17}$$

so that the theory is closed under operator products. This means that R must be a homomorphism from $C \cong \pi_1(G)$ to $\mathrm{Hom}(\pi_1(G), U(1))$. For $C \cong \mathbb{Z}_N$,

$$\mathrm{Hom}(\mathbb{Z}_N, \mathrm{Hom}(\mathbb{Z}_N, U(1))) \cong \mathrm{Hom}(\mathbb{Z}_N, U(1)) \,, \tag{2.18}$$

as each $R \in \mathrm{Hom}(\mathbb{Z}_N, \mathrm{Hom}(\mathbb{Z}_N, U(1)))$ is already uniquely determined by R_{c_1}, the representation corresponding to the generator c_1 of $\mathbb{Z}_N$. On the other hand, for $C \cong \mathbb{Z}_2 \times \mathbb{Z}_2$,

$$\mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))) \cong \mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \times \mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \,, \tag{2.19}$$

as every $R \in \mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)))$ is uniquely determined by $R_{(1,0)}$ and $R_{(0,1)}$. Thus, there should exist different quantisations of the WZW model corresponding to $G = \widetilde{G}/C$, which are labelled by $\mathrm{Hom}(C, U(1))$ (for $C \cong \mathbb{Z}_N$) and by $\mathrm{Hom}(\mathbb{Z}_2\times\mathbb{Z}_2, U(1))\times\mathrm{Hom}(\mathbb{Z}_2\times\mathbb{Z}_2, U(1))$ (for $C \cong \mathbb{Z}_2 \times \mathbb{Z}_2$).

In general, these theories are not invariant under the monodromy corresponding to the analytic continuation of one field about another one. However, as we shall show in the next section, the operator product transforms covariantly with respect to a one-dimensional representation. The appearance of an 'anyonic' representation, describing the monodromy, indicates that the corresponding theories are genuinely braided. The appearance of the braid group is quite

[2] As C is abelian, only one-dimensional representations of the fundamental group appear.

generic for two-dimensional quantum field theories [12, 9, 10]. On the other hand, such theories have traditionally been excluded in Euclidean conformal field theory.
The theories are also not modular invariant in general. However, as we shall show, the modular invariant theory of Felder, Gawedzki and Kupiainen [8] is one of the possible quantisations. It corresponds to a quantisation which is invariant under monodromy, but it is not characterised by this property alone. In fact, we shall show that for $C = \mathbb{Z}_N$ with N even, there exists another monodromy invariant theory (which is then not modular invariant), and for $C = \mathbb{Z}_2 \times \mathbb{Z}_2$, there exist in addition three monodromy invariant quantisations for each of the two different modular invariant solutions.
We should also mention that the above analysis resembles somewhat the treatment in [29], where simple currents were used to construct modular invariant partition functions via an orbifold construction.

From a general point of view, regarding the quantum states as wave-functions on configuration space, one would expect that the different quantisations correspond to the different choices for the 'Aharanov-Bohm' phases in each component of the configuration space. If we insist that the quantum theory should be symmetric under the full loop group (which acts naturally on the configuration space), then the choice for the 'Aharanov-Bohm' phases in the identity component fixes the phases in all other components. We would thus expect that the different quantisations are parametrised by $\mathrm{Hom}(\pi_1(G), U(1))$ as explained in [5]. However, since we are interested in a quantum field theory where states and fields are in one-to-one correspondence, we have the additional constraint that the theory should be closed under the operator product. It is natural to believe that the Aharanov-Bohm phases multiply under the operator product, and then the closure condition implies that the phases of the identity component have to be trivial. We therefore expect that there exists only one quantum field theory which is symmetric under the full loop group, the theory with trivial Aharanov-Bohm phases. This theory, however, is not modular invariant in general as the modular invariant theories do not always satisfy $R_c = id$ for all $c \in C$ [8].
The above theories (2.16) are only symmetric under the identity component of the loop group, and thus, a priori, the phases in the different components are unrelated. The possible theories are then selected by the condition that they are closed under operator products. As the phases multiply, this implies that the different quantisations are parametrised by $\mathrm{Hom}(\pi_1(G), \mathrm{Hom}(\pi_1(G), U(1)))$. Depending on the structure of the fundamental group one or two of these theories are the modular invariant theories of [8]. These theories were selected in [8] by the specific choice [8, eq. (4.9)], relating the Aharanov-Bohm phases in the different components.

3 Monodromy covariance

The key step in the analysis of the monodromy is the calculation of the transformation of the spectrum of L_0 under conjugation by a loop corresponding to $c \in C$. A priori, as described

in Pressley and Segal [28], the action (by conjugation) of the non-trivial loops of G on $\tilde{L}\widetilde{G}$ is well-defined, and thus induces a well-defined action on the (untwisted) Kac-Moody algebra $\hat{\mathfrak{g}} \cong l\mathfrak{g} \oplus \mathbb{R}$, the Lie algebra of $\tilde{L}\widetilde{G}$. On the other hand, a priori a choice has to be made for the definition of the conjugation on $\mathbb{R} \oplus l\mathfrak{g} \oplus \mathbb{R}$, the Lie algebra of the extension $\Pi \triangleright \tilde{L}\widetilde{G}$ of $\tilde{L}\widetilde{G}$ by the rigid rotations Π whose generator is L_0 (see section 4.9 of [28] and the appendix). This ambiguity can be removed by identifying L_0 with the Sugawara expression which is quadratic in $\hat{\mathfrak{g}}$.

The Kac-Moody algebra $\hat{\mathfrak{g}}$ can be described in a modified Cartan-Weyl basis as follows [16]:

$$
\begin{aligned}
[H^i_m, H^j_n] &= km\delta^{ij}\delta_{m,-n} \\
[H^i_m, E^\alpha_n] &= \alpha^i E^\alpha_{m+n} \\
[E^\alpha_m, E^\beta_n] &= \begin{cases} \varepsilon(\alpha,\beta) E^{\alpha+\beta}_{m+n} & \text{if } \alpha+\beta \text{ is a root,} \\ \dfrac{2}{\alpha^2}\left(\alpha \cdot H_{m+n} + km\delta_{m,-n}\right) & \text{if } \alpha = -\beta\ , \\ 0 & \text{otherwise.} \end{cases} \\
[k, E^\alpha_n] &= [k, H^i_n] = 0\,.
\end{aligned}
\tag{3.1}
$$

Here $i = 1, \ldots, r = \text{rank}\ \mathfrak{g}$, α labels the positive roots R^+ of $\mathfrak{g}$, and the horizontal subalgebra (with $n = 0$) is isomorphic to $\mathfrak{g}$.

Upon conjugation with the loop $\theta \in [0, 2\pi] \mapsto exp(\theta c)$, where $exp(2\pi c) \in \mathcal{Z}$, the centre of $\widetilde{G}$, the generators of the Kac-Moody algebra transform as [28]

$$H^i_m \mapsto H^i_m - \delta_{m,0} k \langle c, H^i \rangle\ , \tag{3.2}$$

$$E^\alpha_m \mapsto E^\alpha_{m - \langle \alpha, c\rangle}\ , \tag{3.3}$$

$$k \mapsto k\ , \tag{3.4}$$

where $\langle ., . \rangle$ denotes the Killing-form of the horizontal subalgebra.

The Sugawara expression for L_0 is given as [16]

$$
\begin{aligned}
L_0 = \frac{1}{\beta}\Bigg[&\sum_{i=1}^{r} H^i_0 H^i_0 + \sum_{\alpha\in R^+}\left(E^\alpha_0 E^{-\alpha}_0 + E^{-\alpha}_0 E^\alpha_0\right) \\
&+ 2\sum_{n=1}^{\infty}\left(\sum_{i=1}^{r} H^i_{-n}H^i_n + \sum_{\alpha\in R^+}\left(E^\alpha_{-n}E^{-\alpha}_n + E^{-\alpha}_{-n}E^\alpha_n\right)\right)\Bigg]\ ,
\end{aligned}
\tag{3.5}
$$

where $\beta = 2k + Q_\psi$, and Q_ψ is the quadratic Casimir in the adjoint representation (with highest weight ψ, where ψ is the highest root). Upon conjugation with the loop $\theta \in [0, 2\pi] \mapsto exp(\theta c)$ L_0 becomes

$$L'_0 \;=\; \frac{1}{\beta}\Bigg[\sum_{i=1}^{r}\left(H^i_0 H^i_0 - 2k\langle c, H^i\rangle H^i_0 + k^2 \langle c, H^i\rangle\langle H^i, c\rangle\right)$$

$$
+\sum_{\alpha\in R^+}\left(E^{\alpha}_{-\langle\alpha,c\rangle}E^{-\alpha}_{\langle\alpha,c\rangle}+E^{-\alpha}_{\langle\alpha,c\rangle}E^{\alpha}_{-\langle\alpha,c\rangle}\right)
$$

$$
+2\sum_{n=1}^{\infty}\left(\sum_{i=1}^{r}H^i_{-n}H^i_n+\sum_{\alpha\in R^+}\left(E^{\alpha}_{-n-\langle\alpha,c\rangle}E^{-\alpha}_{n+\langle\alpha,c\rangle}+E^{-\alpha}_{-n+\langle\alpha,c\rangle}E^{\alpha}_{n-\langle\alpha,c\rangle}\right)\right)\Bigg]\,. \qquad (3.6)
$$

We have the identities

$$
-\sum_{i=1}^{r}2\,k\,\langle c,H^i\rangle H^i_0\,|\lambda\rangle=-2\,k\,\langle c,\lambda\rangle\,|\lambda\rangle\,, \qquad (3.7)
$$

where λ is the weight of the state $|\lambda\rangle$ on which L_0' is evaluated, and

$$
\sum_{i=1}^{r}k^2\langle c,H^i\rangle\langle H^i,c\rangle=k^2\langle c,c\rangle\,. \qquad (3.8)
$$

Furthermore, we can choose (without loss of generality) the set of positive roots, R^+, such that $\langle\alpha,c\rangle\geq 0$ for all $\alpha\in R^+$. Then we can rewrite $L_0'|\lambda\rangle$ as

$$
\begin{aligned}
&L_0|\lambda\rangle+\frac{1}{\beta}\Bigg[-2k\langle c,\lambda\rangle+k^2\langle c,c\rangle\\
&\quad+\sum_{\alpha\in R^+}\left(2\sum_{n=1}^{\langle\alpha,c\rangle-1}[E^{-\alpha}_{-n+\langle\alpha,c\rangle},E^{\alpha}_{n-\langle\alpha,c\rangle}]+[E^{-\alpha}_0,E^{\alpha}_0]+[E^{-\alpha}_{\langle\alpha,c\rangle},E^{\alpha}_{-\langle\alpha,c\rangle}]\right)|\lambda\rangle\Bigg]\\
&=\;L_0|\lambda\rangle+\frac{1}{\beta}\left[-2k\langle c,\lambda\rangle+k^2\langle c,c\rangle+2\sum_{\alpha\in R^+}\left(\sum_{n=0}^{\langle\alpha,c\rangle-1}[E^{-\alpha}_{-n+\langle\alpha,c\rangle},E^{\alpha}_{n-\langle\alpha,c\rangle}]-\frac{k}{\alpha^2}\langle\alpha,c\rangle\right)|\lambda\rangle\right]\\
&=\;L_0|\lambda\rangle+\frac{1}{\beta}\Bigg[-2k\langle c,\lambda\rangle+k^2\langle c,c\rangle\\
&\quad+\sum_{\alpha\in R^+}\left(\sum_{n=0}^{\langle\alpha,c\rangle-1}\left\{-\frac{4}{\alpha^2}\langle\alpha,\lambda\rangle+\frac{4}{\alpha^2}k(-n+\langle\alpha,c\rangle)\right\}-\frac{2k}{\alpha^2}\langle\alpha,c\rangle\right)\Bigg]\\
&=\;L_0|\lambda\rangle+\frac{1}{\beta}\left[-2k\langle c,\lambda\rangle+k^2\langle c,c\rangle+\sum_{\alpha\in R^+}\left\{-\frac{4}{\alpha^2}\langle\alpha,\lambda\rangle\langle\alpha,c\rangle+\frac{4}{\alpha^2}k\left(\sum_{l=1}^{\langle\alpha,c\rangle}l\right)-\frac{2k}{\alpha^2}\langle\alpha,c\rangle\right)\right]\\
&=\;L_0|\lambda\rangle+\frac{1}{\beta}\left[-2k\langle c,\lambda\rangle+k^2\langle c,c\rangle+\sum_{\alpha\in R^+}\left\{-\frac{4}{\alpha^2}\langle\alpha,\lambda\rangle\langle\alpha,c\rangle+\frac{2k}{\alpha^2}\langle\alpha,c\rangle\langle c,\alpha\rangle\right\}\right]\,.
\end{aligned} \qquad (3.9)
$$

Finally, we can use the identity (see *e.g.* [17])

$$
\sum_{\alpha\in R^+}\frac{4}{\alpha^2}\;|\alpha\rangle\langle\alpha|=Q_\psi\mathbb{1}_r\,, \qquad (3.10)
$$

where $\mathbb{1}_r$ is the unit matrix in the space of rank $\mathfrak{g}$ matrices, to conclude that

$$L_0' \,|\lambda\rangle = L_0 \,|\lambda\rangle - \langle c, \lambda\rangle \,|\lambda\rangle + \frac{1}{2}\, k\, \langle c, c\rangle \,|\lambda\rangle \,, \tag{3.11}$$

as $\beta = 2k + Q_\psi$.

If c is a coroot, the conjugation corresponds to a transformation in the affine Weyl group $\overline{\mathcal{W}}(\mathfrak{g})$, and the induced transformation on weights coincides with the formula given in [16]. We should also mention that essentially this formula has been derived in [13] for $\mathfrak{g} = a_n$, and, in a different context, in [11].

We are now in the position to analyse the monodromy properties of the different quantisations (2.16). Recall that for any weight $\overline{\lambda}$ in a subrepresentation of the tensor product of two representations with highest weight $\overline{\lambda}_1$ and $\overline{\lambda}_2$, respectively, and for any $exp(2\pi c) \in \mathcal{Z}$, we have

$$\langle c, \overline{\lambda}\rangle = \langle c, \overline{\lambda}_1\rangle + \langle c, \overline{\lambda}_2\rangle \pmod 1 \,. \tag{3.12}$$

Furthermore, the product of two states $(\lambda_1, c_1(\overline{\lambda}_1))$ and $(\lambda_2, c_2(\overline{\lambda}_2))$ in (2.16) is in the sector $(\lambda, (c_1 + c_2)(\overline{\lambda}))$. As the L_0 spectrum is the same for a representation and its conjugate, upon rotating the field corresponding to $(\lambda_1, c_1(\overline{\lambda}_1))$ by 2π about $(\lambda_2, c_2(\overline{\lambda}_2))$, the three-point function changes by

$$R(\lambda_1, c_1; \lambda_2, c_2) = exp\Big\{2\pi\Big(-\langle\overline{\lambda}_1, c_2\rangle - \langle\overline{\lambda}_2, c_1\rangle + k\langle c_1, c_2\rangle\Big)\Big\} \,, \tag{3.13}$$

which is independent of λ. This demonstrates that the fields of (2.16) are covariant under monodromy with respect to a one-dimensional representation which is given by (3.13).

4 Additional monodromy invariant theories

We want to analyse now, how many monodromy invariant quantisations exist. Recall from section 2 that the different quantisations are of the form

$$\mathcal{H}^R_G := \sum_{c\in C} \sum_{j\in R_c} \mathcal{H}_j \otimes \mathcal{H}_{c(\bar{j})} \,, \tag{4.1}$$

where $G = \widetilde{G}/C$ and R_c is an equivalence class of positive energy representations of $\hat{\mathfrak{g}}$ corresponding to a representation of C (which we also denote by $R_c : C \to U(1)$). In order for the theory to be closed under composition, the assignment of R_c to c must respect the group structure of C, *i.e.* R must be an element of $\mathrm{Hom}(C, \mathrm{Hom}(C, U(1)))$. The quantisation is invariant under monodromy, if $R(\lambda_1, c_1; \lambda_2, c_2) = 1$ for all $(\lambda_i, c_i(\overline{\lambda}_i))$ in (4.1). Let us consider the two different cases for the structure of C separately.

4.1 The case $C = \mathbb{Z}_N$.

Let c denote a cyclic generator of C, where the order of c is N. Because of the representation property of R, all $R_{c'}$ are uniquely determined, once R_c is fixed. Let $\lambda \in R_c$. By requiring monodromy invariance for the product of (λ, c) with itself, we find using (3.13)

$$0 = 2\left(-\langle\overline{\lambda}, c\rangle + \frac{1}{2}k\langle c, c\rangle\right) \pmod 1 . \tag{4.2}$$

As C has order N, any possible weight $\overline{\lambda}$ has to satisfy

$$N\langle\overline{\lambda}, c\rangle = 0 \pmod 1 , \tag{4.3}$$

and any linear functional, satisfying (4.3), corresponds to a class of possible weights. The quantisation condition

$$\frac{1}{2}Nk\langle c, c\rangle \in \mathbb{Z} \tag{4.4}$$

thus guarantees that there exists $\overline{\lambda}$ such that

$$-\langle\overline{\lambda}, c\rangle + \frac{1}{2}k\langle c, c\rangle = 0 \pmod 1 . \tag{4.5}$$

In this case, the difference of the L_0 eigenvalues of the representation corresponding to λ and to $c(\overline{\lambda})$ is integral because of (3.11) and since a representation and its conjugate have the same L_0 spectrum. The same is true for all other sectors, since

$$-\langle m\overline{\lambda}, mc\rangle + \frac{1}{2}k\langle mc, mc\rangle = m^2\left(-\langle\overline{\lambda}, c\rangle + \frac{1}{2}k\langle c, c\rangle\right) = 0 \pmod 1 . \tag{4.6}$$

Thus this solution corresponds to the unique modular invariant theory of Felder *et.al.* [8].

Because of the consistency condition (4.3) and the quantisation condition (4.4), there exists a solution $\overline{\lambda}_1$, satisfying

$$-\langle\overline{\lambda}_1, c\rangle + \frac{1}{2}k\langle c, c\rangle = \frac{1}{2} \pmod 1 , \tag{4.7}$$

only if N is even. On the other hand, for even N, $\overline{\lambda}_1$ is a possible weight as it corresponds to a representation in the equivalence class of the one-dimensional representation of the centre

$$R(w) = -e^{\pi i k\langle w, w\rangle} , \tag{4.8}$$

where $w \in C$. To check that the corresponding theory is monodromy invariant, we observe that

$$-\langle m\overline{\lambda}_1, nc\rangle - \langle n\overline{\lambda}_1, mc\rangle + k\langle mc, nc\rangle = mn\left(-2\langle\overline{\lambda}_1, c\rangle + k\langle c, c\rangle\right) \in \mathbb{Z} . \tag{4.9}$$

Thus all sectors are relatively monodromy invariant. It is clear that this additional monodromy invariant theory is not modular invariant, as the partition function is not invariant under $\tau \mapsto \tau + 1$.

4.2 The case $C = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Let $c_1 = (1,0)$ and $c_2 = (0,1)$ be the two generators of $C = \mathbb{Z}_2 \times \mathbb{Z}_2$, and denote by λ_1 and λ_2 the corresponding weights in (4.1). We have to check, case by case, the conditions implied by monodromy invariance..

(i) $(1,0) \times (1,0) = (0,0)$. The requirement is

$$-2\langle\overline{\lambda}_1, c_1\rangle + k\langle c_1, c_1\rangle = 0 \pmod 1 . \tag{4.10}$$

As $2c_1$ is a coroot, $2\langle\overline{\lambda}_1, c_1\rangle \in \mathbb{Z}$, and the condition is

$$k\langle c_1, c_1\rangle \in \mathbb{Z} , \tag{4.11}$$

which is one of the quantisation conditions.

(ii) $(0,1) \times (0,1) = (0,0)$. An identical reasoning gives

$$k\langle c_2, c_2\rangle \in \mathbb{Z} , \tag{4.12}$$

which is again one of the quantisation conditions.

(iii) $(1,1) \times (1,1) = (0,0)$. Similarly we find

$$k\langle c_1 + c_2, c_1 + c_2\rangle \in \mathbb{Z} , \tag{4.13}$$

which follows from one of the quantisation conditions.

(iv) $(1,0) \times (0,1) = (1,1)$. Using (3.13), we find

$$-\langle\overline{\lambda}_1, c_2\rangle - \langle\overline{\lambda}_2, c_1\rangle + k\langle c_1, c_2\rangle \in \mathbb{Z} . \tag{4.14}$$

As explained in [8], there are two modular invariant theories. One is characterised by

$$\begin{aligned}
\langle\overline{\lambda}_1, c_1\rangle &= \frac{k}{2}\langle c_1, c_1\rangle + \frac{n}{2} \pmod 1 \\
\langle\overline{\lambda}_1, c_2\rangle &= \frac{k}{2}\langle c_1, c_2\rangle \pmod 1 \\
\langle\overline{\lambda}_2, c_1\rangle &= \frac{k}{2}\langle c_2, c_1\rangle \pmod 1 \\
\langle\overline{\lambda}_2, c_2\rangle &= \frac{k}{2}\langle c_2, c_2\rangle + \frac{m}{2} \pmod 1 ,
\end{aligned} \tag{4.15}$$

where $m = n = 0$, and the other is characterised by

$$\begin{aligned}
\langle \overline{\lambda}_1, c_1 \rangle &= \frac{k}{2}\langle c_1, c_1 \rangle + \frac{n}{2} \pmod 1 \\
\langle \overline{\lambda}_1, c_2 \rangle &= \frac{k}{2}\langle c_1, c_2 \rangle + \frac{1}{2} \pmod 1 \\
\langle \overline{\lambda}_2, c_1 \rangle &= \frac{k}{2}\langle c_2, c_1 \rangle + \frac{1}{2} \pmod 1 \\
\langle \overline{\lambda}_2, c_2 \rangle &= \frac{k}{2}\langle c_2, c_2 \rangle + \frac{m}{2} \pmod 1 \,,
\end{aligned} \tag{4.16}$$

where, again, $m = n = 0$.
To each of the two solutions, there exist additional monodromy invariant solutions given by (4.15) and (4.16), respectively, with $(n = 1, m = 0)$, $(n = 0, m = 1)$ and $(n = 1, m = 1)$. It is immediate that they satisfy all constraints. It is clear that these theories are not modular invariant, as the partition functions are not invariant under $\tau \mapsto \tau + 1$.

5 Examples

In this section we shall give a few non-trivial examples, exhibiting the additional (monodromy invariant) quantisations. The simplest example occurs for $G = SO(3)$ and was already pointed out in [5].

5.1 $G = SO(3)$.

The first homotopy group of $SO(3)$ is $\pi_1(SO(3)) = \mathbb{Z}_2$, and the generator of $\pi_1(SO(3))$, written as an element of $\mathfrak{h}$, is $c = \sqrt{2}/2$. (Recall, that $E^{\pm}$ are $\pm\sqrt{2}$ in this notation.) The quantisation condition requires k to be even, as $\langle c, c \rangle = 1/2$ (see (2.3)).
The outer automorphism corresponding to c acts on representations by mapping the representation with spin j to the one with spin $c(j) = k/2 - j$. There are two cases to consider

(i) $k = 4n$. The modular invariant theory is given by

$$\begin{aligned}
\mathcal{H}_0 &= \Big[\big([0] \otimes [0]\big) \oplus \big[1] \otimes [1]\big) \oplus \big[2] \otimes [2]\big) \oplus \ldots\Big] \\
&\quad \oplus \Big[\big([0] \otimes [k/2]\big) \oplus \big([1] \otimes [k/2 - 1]\big) \oplus \ldots\Big] \,.
\end{aligned}$$

(ii) $k = 2n + 2$. The modular invariant theory is given by

$$\begin{aligned}
\mathcal{H}_1 &= \Big[\big([0] \otimes [0]\big) \oplus \big([1] \otimes [1]\big) \oplus \big([2] \otimes [2]\big) \oplus \ldots\Big] \\
&\quad \oplus \Big[\big([1/2] \otimes [k/2 - 1/2]\big) \oplus \big([3/2] \otimes [k/2 - 3/2]\big) \oplus \ldots\Big] \,.
\end{aligned}$$

The additional quantisation which is also monodromy invariant is for $k = 4n$, $\mathcal{H}_1$, and for $k = 2n+2$, $\mathcal{H}_0$. To check that these theories are indeed monodromy invariant, we note that the conformal weight of the lowest energy space of the representation $[j]$ is

$$L_0([j]) = \frac{j(j+1)}{k+2}\,, \tag{5.1}$$

and that

$$L_0([c(j)]) - L_0([j]) = \frac{k}{4} - j\,. \tag{5.2}$$

This agrees with (3.11), as $\langle c, j\rangle = j$, and the claimed monodromy invariance is easily verified.

5.2 Quotient groups of $G = SU(4)$.

The centre of $SU(4)$ is $\mathcal{Z} \cong \mathbb{Z}_4$, and the group of central elements is generated by $c = \lambda_1$, the fundamental weight corresponding to the first root (for the notation see for example [20]). The central elements act (as outer automorphisms) on the representations $[l,m,n]$, written in the Dynkin basis, as

$$\begin{array}{rcl} c([l,m,n]) & = & [m,n,k-l-m-n]\,, \\ c^2([l,m,n]) & = & [n,k-l-m-n,l]\,, \\ c^3([l,m,n]) & = & [k-l-m-n,l,m]\,. \end{array} \tag{5.3}$$

The conformal weight of the lowest energy space of the representation $[l,m,n]$ is

$$L_0([l,m,n]) = \Big(\frac{1}{4}(3l^2+4m^2+3n^2+4lm+4mn+2ln) + (3l+4m+3n)\Big)/(2(k+4))\,. \tag{5.4}$$

Using these explicit formulae, it is easy to check that (3.11) holds.

The centre contains the two different subgroups $\mathbb{Z}_2$ and $\mathbb{Z}_4$, and thus there are two different quotient groups whose simply connected covering group is $SU(4)$. Let us discuss the two cases in turn.

(i) $G = SU(4)/\mathbb{Z}_4$. The quantisation condition is $k \in 2\mathbb{Z}$, as $\langle c,c\rangle = 3/4$ and $N = 4$ (see (2.3)). We have to consider the four cases $k = 8p + 2s$, where $p \in \mathbb{Z}$ and $s = 0,1,2,3$.

- $k = 8p$. The modular invariant partition function is

$$\begin{array}{rcl} \mathcal{H}_0 & = & \Big[\Big([0,0,0]\otimes[0,0,0]\Big) \oplus \Big([1,0,1]\otimes[1,0,1]\Big) \oplus \ldots\Big] \\ & & \oplus\Big[\Big([0,0,0]\otimes[0,0,k]\Big) \oplus \Big([1,0,1]\otimes[0,1,k-2]\Big) \oplus \ldots\Big] \\ & & \oplus\Big[\Big([0,0,0]\otimes[0,k,0]\Big) \oplus \Big([1,0,1]\otimes[1,k-2,1]\Big) \oplus \ldots\Big] \\ & & \oplus\Big[\Big([0,0,0]\otimes[k,0,0]\Big) \oplus \Big([1,0,1]\otimes[k-2,1,0]\Big) \oplus \ldots\Big]\,. \end{array}$$

- $k = 8p + 2$. The modular invariant partition function is

$$\begin{aligned}\mathcal{H}_1 \;=\;& \Big[\Big([0,0,0]\otimes[0,0,0]\Big)\oplus\Big([1,0,1]\otimes[1,0,1]\Big)\oplus\ldots\Big]\\ &\oplus\Big[\Big([1,0,0]\otimes[0,0,k-1]\Big)\oplus\Big([0,1,1]\otimes[1,1,k-2]\Big)\oplus\ldots\Big]\\ &\oplus\Big[\Big([0,1,0]\otimes[0,k-1,0]\Big)\oplus\Big([1,1,1]\otimes[1,k-3,1]\Big)\oplus\ldots\Big]\\ &\oplus\Big[\Big([0,0,1]\otimes[k-1,0,0]\Big)\oplus\Big([1,1,0]\otimes[k-2,1,1]\Big)\oplus\ldots\Big].\end{aligned}$$

- $k = 8p + 4$. The modular invariant partition function is

$$\begin{aligned}\mathcal{H}_2 \;=\;& \Big[\Big([0,0,0]\otimes[0,0,0]\Big)\oplus\Big([1,0,1]\otimes[1,0,1]\Big)\oplus\ldots\Big]\\ &\oplus\Big[\Big([0,1,0]\otimes[1,0,k-1]\Big)\oplus\Big([2,0,0]\otimes[0,0,k-2]\Big)\oplus\ldots\Big]\\ &\oplus\Big[\Big([0,0,0]\otimes[0,k,0]\Big)\oplus\Big([1,0,1]\otimes[1,k-2,1]\Big)\oplus\ldots\Big]\\ &\oplus\Big[\Big([0,1,0]\otimes[k-1,0,1]\Big)\oplus\Big([0,0,2]\otimes[k-2,0,0]\Big)\oplus\ldots\Big].\end{aligned}$$

- $k = 8p + 6$. The modular invariant partition function is

$$\begin{aligned}\mathcal{H}_3 \;=\;& \Big[\Big([0,0,0]\otimes[0,0,0]\Big)\oplus\Big([1,0,1]\otimes[1,0,1]\Big)\oplus\ldots\Big]\\ &\oplus\Big[\Big([0,0,1]\otimes[0,1,k-1]\Big)\oplus\Big([1,1,0]\otimes[1,0,k-2]\Big)\oplus\ldots\Big]\\ &\oplus\Big[\Big([0,1,0]\otimes[0,k-1,0]\Big)\oplus\Big([1,1,1]\otimes[1,k-3,1]\Big)\oplus\ldots\Big]\\ &\oplus\Big[\Big([1,0,0]\otimes[k-1,1,0]\Big)\oplus\Big([0,1,1]\otimes[k-2,0,1]\Big)\oplus\ldots\Big].\end{aligned}$$

The different quantisations are for each even k, $\mathcal{H}_0, \ldots, \mathcal{H}_3$. The additional monodromy invariant quantisation is, for $k = 8p$ $\mathcal{H}_2$, for $k = 8p+2$ $\mathcal{H}_3$, for $k = 8p+4$ $\mathcal{H}_0$, and for $k = 8p+6$ $\mathcal{H}_1$. Using (5.4), it is easy to see that these theories are indeed monodromy invariant.

(ii) $G = SU(4)/\mathbb{Z}_2$. The generator of $\mathbb{Z}_2$ is $d = 2c$. The quantisation condition is $k \in \mathbb{Z}$, as $\langle d, d\rangle = 4$ and $N = 2$ (see (2.3)). We have to consider two cases, k even and k odd.

- For k even, the modular invariant theory is given by

$$\begin{aligned}\mathcal{H}_0 \;=\;& \Big[\Big([0,0,0]\otimes[0,0,0]\Big)\oplus\Big([0,1,0]\otimes[0,1,0]\Big)\oplus\ldots\Big]\\ &\oplus\Big[\Big([0,0,0]\otimes[0,k,0]\Big)\oplus\Big([0,1,0]\otimes[0,k-1,0]\Big)\oplus\ldots\Big].\end{aligned}$$

- For odd k, the modular invariant theory is given by

$$\begin{aligned}\mathcal{H}_1 \;=\;& \Big[\Big([0,0,0]\otimes[0,0,0]\Big)\oplus\Big([0,1,0]\otimes[0,1,0]\Big)\oplus\ldots\Big]\\ &\oplus\Big[\Big([1,0,0]\otimes[0,k-1,1]\Big)\oplus\Big([0,0,1]\otimes[1,k-1,0]\Big)\oplus\ldots\Big].\end{aligned}$$

Again, as before, the additional quantisation which is also monodromy invariant is for k even $\mathcal{H}_1$, and for k odd $\mathcal{H}_0$. As before, the monodromy invariance can be easily checked using (5.4).

5.3 $G = SO(8)/\mathbb{Z}_2 \times \mathbb{Z}_2$.

The centre of $G = SO(8)$ is $\mathcal{Z} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and it is generated by $(1,0) = \lambda_1$, $(0,1) = \lambda_3$ and $(1,1) = \lambda_4$, where λ_i is the fundamental weight corresponding to the ith root. The central elements act (as outer automorphisms) on the representations $[r_1, r_2, r_3, r_4]$, written in the Dynkin basis, as

$$\begin{aligned} c_{(1,0)}([r_1,r_2,r_3,r_4]) &= [k-r_1-2r_2-r_3-r_4, r_2, r_4, r_3]\,, \\ c_{(0,1)}([r_1,r_2,r_3,r_4]) &= [r_4, r_2, k-r_1-2r_2-r_3-r_4, r_1]\,, \\ c_{(1,1)}([r_1,r_2,r_3,r_4]) &= [r_3, r_2, r_1, k-r_1-2r_2-r_3-r_4]\,. \end{aligned} \tag{5.5}$$

The conformal weight of the lowest energy space of the representation $[r_1, r_2, r_3, r_4]$ is

$$\begin{aligned} L_0([r_1,r_2,r_3,r_4]) = \Big(& r_1^2+2r_2^2+r_3^2+r_4^2+2r_2(r_1+r_3+r_4)+r_1r_3+r_1r_4+r_3r_4 \\ & +6r_1+10r_2+6r_3+6r_4\Big)/\left(2(k+6)\right)\,. \end{aligned} \tag{5.6}$$

It is easy to see that the formula for the difference of the L_0 spectrum (3.11) holds.

The quantisation condition is $k \in 2\mathbb{Z}$, and for each allowed k there are two different modular invariant theories. They are explicitly given as

$$\begin{aligned} \mathcal{H}_0 = & \oplus\Big[\Big([r_1,r_2,r_3,r_4]\otimes[r_1,r_2,r_3,r_4]\Big)\Big| r_{13}\text{ even}; r_{14}\text{ even}; r_{34}\text{ even}\Big] \\ & \oplus\Big[\Big([r_1,r_2,r_3,r_4]\otimes c_{(1,0)}([r_1,r_2,r_3,r_4])\Big)\Big| r_{34}\text{ even}; r_{13}+\frac{k}{2}\text{ even}; r_{14}+\frac{k}{2}\text{ even}\Big] \\ & \oplus\Big[\Big([r_1,r_2,r_3,r_4]\otimes c_{(0,1)}([r_1,r_2,r_3,r_4])\Big)\Big| r_{14}\text{ even}; r_{13}+\frac{k}{2}\text{ even}; r_{34}+\frac{k}{2}\text{ even}\Big] \\ & \oplus\Big[\Big([r_1,r_2,r_3,r_4]\otimes c_{(1,1)}([r_1,r_2,r_3,r_4])\Big)\Big| r_{13}\text{ even}; r_{14}+\frac{k}{2}\text{ even}; r_{34}+\frac{k}{2}\text{ even}\Big]\,, \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \mathcal{H}_1 = & \oplus\Big[\Big([r_1,r_2,r_3,r_4]\otimes[r_1,r_2,r_3,r_4]\Big)\Big| r_{13}\text{ even}; r_{14}\text{ even}; r_{34}\text{ even}\Big] \\ & \oplus\Big[\Big([r_1,r_2,r_3,r_4]\otimes c_{(1,0)}([r_1,r_2,r_3,r_4])\Big)\Big| r_{34}\text{ even}; r_{13}+\frac{k}{2}\text{ odd}; r_{14}+\frac{k}{2}\text{ odd}\Big] \\ & \oplus\Big[\Big([r_1,r_2,r_3,r_4]\otimes c_{(0,1)}([r_1,r_2,r_3,r_4])\Big)\Big| r_{14}\text{ even}; r_{13}+\frac{k}{2}\text{ odd}; r_{34}+\frac{k}{2}\text{ odd}\Big] \\ & \oplus\Big[\Big([r_1,r_2,r_3,r_4]\otimes c_{(1,1)}([r_1,r_2,r_3,r_4])\Big)\Big| r_{13}\text{ even}; r_{14}+\frac{k}{2}\text{ odd}; r_{34}+\frac{k}{2}\text{ odd}\Big]\,, \end{aligned} \tag{5.8}$$

where $r_{ij} = r_i + r_j$, and only those representations appear which are positive energy, *i.e.* satisfy $r_1 + 2r_2 + r_3 + r_4 \leq k$. To shorten the notation, let us describe in the following the theories by 4×3 matrices with entries e (even) and o (odd), where the first row corresponds

to (r_{13}, r_{14}, r_{34}) in the identity sector, the second row corresponds to $(r_{34}, r_{13}+k/2, r_{14}+k/2)$ in the $c_{(1,0)}$ sector, and so on. For example

$$\mathcal{H}_0 = \begin{pmatrix} e & e & e \\ e & e & e \\ e & e & e \\ e & e & e \end{pmatrix} \qquad \mathcal{H}_1 = \begin{pmatrix} e & e & e \\ e & o & o \\ e & o & o \\ e & o & o \end{pmatrix} . \tag{5.9}$$

The additional monodromy invariant theories corresponding to $\mathcal{H}_0$ are then given as

$$\mathcal{H}_0^{(1)} = \begin{pmatrix} e & e & e \\ o & o & e \\ e & e & e \\ o & e & o \end{pmatrix} \qquad \mathcal{H}_0^{(2)} = \begin{pmatrix} e & e & e \\ e & e & e \\ o & o & e \\ o & o & e \end{pmatrix} \qquad \mathcal{H}_0^{(3)} = \begin{pmatrix} e & e & e \\ o & o & e \\ o & o & e \\ e & o & o \end{pmatrix} , \tag{5.10}$$

and the additional monodromy invariant theories corresponding to $\mathcal{H}_1$ are

$$\mathcal{H}_1^{(1)} = \begin{pmatrix} e & e & e \\ o & e & o \\ e & o & o \\ o & o & e \end{pmatrix} \qquad \mathcal{H}_1^{(2)} = \begin{pmatrix} e & e & e \\ e & o & o \\ o & e & o \\ o & e & o \end{pmatrix} \qquad \mathcal{H}_1^{(3)} = \begin{pmatrix} e & e & e \\ o & e & o \\ o & e & o \\ e & e & e \end{pmatrix} . \tag{5.11}$$

From the formulae given above, it is easy to see that these theories are indeed monodromy invariant. Furthermore, there are another 8 quantisations which are not monodromy invariant

$$\mathcal{H}_9 = \begin{pmatrix} e & e & e \\ e & e & e \\ e & o & o \\ o & e & o \end{pmatrix} \quad \mathcal{H}_{10} = \begin{pmatrix} e & e & e \\ e & e & e \\ o & e & o \\ e & o & o \end{pmatrix} \quad \mathcal{H}_{11} = \begin{pmatrix} e & e & e \\ e & o & o \\ e & e & e \\ o & o & e \end{pmatrix} \quad \mathcal{H}_{12} = \begin{pmatrix} e & e & e \\ e & o & o \\ o & o & e \\ e & e & e \end{pmatrix} ,$$

$$\mathcal{H}_{13} = \begin{pmatrix} e & e & e \\ o & e & o \\ e & e & e \\ e & o & o \end{pmatrix} \quad \mathcal{H}_{14} = \begin{pmatrix} e & e & e \\ o & e & o \\ o & o & e \\ o & e & o \end{pmatrix} \quad \mathcal{H}_{15} = \begin{pmatrix} e & e & e \\ o & o & e \\ o & e & o \\ o & o & e \end{pmatrix} \quad \mathcal{H}_{16} = \begin{pmatrix} e & e & e \\ o & o & e \\ e & o & o \\ e & e & e \end{pmatrix} .$$

6 Conclusions

We have analysed the different quantisations of the WZW model corresponding to a in general non-simply connected group G, and we have found that they are parametrised by $\mathrm{Hom}(\pi_1(G), \mathrm{Hom}(\pi_1(G), U(1)))$. In general the different quantisations are genuinely braided and neither monodromy nor modular invariant. However, all the modular invariant theories of Felder *et.al.* [8] are contained among them. Furthermore, for $\pi_1(G) \cong \mathbb{Z}_N$ with N even, there is another monodromy invariant theory, and for $\pi_1(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ there are another 6 monodromy invariant quantisations.

It might be hoped that the different quantisations of the WZW model are characterised by the property to be *modular covariant*. This would mean that it should be possible to define amplitudes for these theories on arbitrary Riemann surfaces. These amplitudes would not be invariant under the action of the modular group. However, they would transform with respect to a (one-dimensional) representation, and therefore the corresponding probabilities would be invariant. (The amplitudes would be rather similar to the amplitudes of a chiral theory, but here they would correspond to the whole theory.) A first indication that this might be the case is the fact that the amplitudes transform with respect to a one-dimensional representation of the monodromy group, as shown in section 3.

Braided conformal field theories similar to the ones discussed in this paper have already appeared in [25].[3] There the conformal limit (in the sector $w = 0$) of the $\phi_{3,1}$ (integrable) perturbation of $\mathcal{M}_{3,5}$ was found to be the theory

$$\mathcal{H}_{3,5}^{3,1} = (\mathcal{H}_{1,1} \otimes \mathcal{H}_{1,1}) \oplus (\mathcal{H}_{3,1} \otimes \mathcal{H}_{3,1}) \oplus (\mathcal{H}_{2,1} \otimes \mathcal{H}_{3,1}) \oplus (\mathcal{H}_{4,1} \otimes \mathcal{H}_{1,1}) \tag{5.12}$$

which is genuinely braided. (Our notation follows, for example, [21].) In this context our analysis seems to suggest that the conformal limit of the different massive perturbations of a conformal field theory correspond to different quantisations and global structures of the underlying conformal theory. It would be interesting to check this conjecture by analysing the conformal limit of massive perturbations of WZW models.

The analysis of the global properties of the WZW models might also be relevant for a better understanding of the global issues of (abelian) T-duality in WZW models [23, 1, 2, 27]. In particular, similar techniques might be used to analyse in which way the duality transformation depends on the global topological properties of the target space group. This is currently work in progress.

Appendix

In this appendix we want to establish some notation about the affine Weyl group. This follows essentially [28]. We shall also give a simple geometrical proof for the observation of Olive and Turok [26] about the symmetries of the affine Dynkin diagram.

A The affine Weyl group

To define the affine Weyl group let us consider the semidirect product of the loop group LG of G by the unit circle Π

$$\Pi \triangleright LG \,, \tag{A.1}$$

[3] I thank G. Watts for drawing my attention to this reference.

where Π acts on LG by rigidly rotating the loops. The introduction of Π is completely analogous to the introduction of the additional generator $d = -L_0$ (see *e.g.* [16]) in the usual description of the affine root system. That is, we can identify

$$\lambda \in \Pi \longleftrightarrow \lambda^{L_0}\,, \tag{A.2}$$

which acts on LG by rigidly rotating the loop, *i.e.* $\lambda.f(t) = f(\lambda t)$.
A maximal abelian subgroup of $\Pi \triangleright LG$ is $(\Pi \times H)$, where H is a maximal torus in G. We can then define the affine Weyl group, analogously to the definition of the Weyl group of G, as

$$\overline{\mathcal{W}}(G) = N(\Pi \times H)/(\Pi \times H)\,, \tag{A.3}$$

where $N(\Pi \times H)$ is the normaliser of $(\Pi \times H)$ in $\Pi \triangleright LG$.
As has been shown in [28], $\overline{\mathcal{W}}(G)$ is the semidirect product of the coweight lattice $\check{H}$ of G by $\mathcal{W}$, the Weyl group of G. Here, the coweight lattice is the lattice of all homomorphisms

$$\Pi \rightarrow H\,. \tag{A.4}$$

For a simply connected group, the coweight lattice is generated by the coroots, and thus

$$\overline{\mathcal{W}}(\widetilde{G}) = \overline{\mathcal{W}}(\mathfrak{g})\,, \tag{A.5}$$

where the right hand side denotes the usual affine Weyl group, associated to the affine Lie algebra $\hat{\mathfrak{g}}$ (see *e.g.* [16]). However, in general, the affine Weyl group $\overline{\mathcal{W}}(G)$ is larger and depends on the group G, rather than on the algebra alone.
To be more specific, let us calculate the action of a coweight $\phi : \Pi \rightarrow H$ on the maximal abelian subgroup $(\Pi \times H)$ of $\Pi \triangleright LG$. Denote a typical element of $(\Pi \times H)$ by (u, h). Then, as $\phi \in LG$,

$$\begin{aligned}(1, \phi(t))^{-1}.(u, h).(1, \phi(t)) &= (1, \phi(t))^{-1}.(u, \phi(ut)h) \\ &= (u, \phi(u)h)\,.\end{aligned} \tag{A.6}$$

(This demonstrates, in particular, that $\check{H}$ is in the normaliser of $(\Pi \times H)$.) The action induces an action on roots, where roots are linear maps

$$(\Pi \times H) \rightarrow \Pi\,. \tag{A.7}$$

They are specified by $(n, \alpha) \in \mathbb{Z} \times H^*$, where

$$(n, \alpha)\Big((u, h)\Big) = u^n \alpha(h)\,. \tag{A.8}$$

The induced action of $\phi \in \check{H}$ on $(\Pi \times H)$ is given as

$$\begin{aligned}\Big(\phi(n, \alpha)\Big)(u, h) &= (n, \alpha)\Big((1, \phi)^{-1}.(u, h).(1, \phi)\Big) \\ &= u^n \alpha(\phi(u))\alpha(h) \\ &= (n + \alpha(\phi), \alpha)\Big((u, h)\Big)\,.\end{aligned} \tag{A.9}$$

Here, we have identified $\phi \in \check{H}$ with $\phi \in \mathfrak{h}$ by $\phi(t = e^\theta) = exp(\theta\phi)$. Then

$$\alpha(\phi(t)) = exp(\theta\alpha(\phi)) = t^{\alpha(\phi)} . \tag{A.10}$$

For coroots, the action in (A.9) agrees with the formula in [16].

So far we have only considered the loop group LG, and not its central extension $\tilde{L}G$ by Π. The Lie algebra of this central extension is the affine Kac-Moody algebra $\hat{\mathfrak{g}}$, defined in section 3. For simply connected G, the consistency condition on k is that it is integral, and in the general case it is

$$k\langle c_1, c_2\rangle \in \mathbb{Z} , \tag{A.11}$$

where c_i are arbitrary coweights [28, Section 4.6]. This condition is stronger than the quantisation conditions of section 2, and, in particular, it suggests that one should *not* regard the states of the quantum theory as sections in the $U(1)$ bundle $\tilde{L}G$ over LG. This is also in accordance with the fact that the theories we are considering here are only invariant under the identity component of the loop group, whereas theories whose states are sections in $\tilde{L}G$ would be symmetric under the full loop group.
The action of coweights on $\tilde{L}\widetilde{G}$ induces a unique action on the Kac-Moody algebra $\hat{\mathfrak{g}}$ [28, Lemma (4.6.5)], and we have given an explicit formula for this action in section 3. We can also (re)introduce the extension of $\tilde{L}\widetilde{G}$ by the rigid rotations Π. The action of coroots on the Lie algebra of this extension, $\mathbb{R} \oplus l\mathfrak{g} \oplus \mathbb{R}$, is then uniquely determined, but the action of coweights (which are not coroots) involves a choice. We can fix this choice (canonically) by identifying $-d = L_0$ with the Sugawara expression as we have done in section 3.

B Symmetries of the affine Dynkin diagram

Let Φ and $\overline{\Phi}$ be the root systems of $\mathfrak{g}$ and $\hat{\mathfrak{g}}$, respectively, and denote by $\mathcal{W}$ and $\overline{\mathcal{W}}$ the corresponding Weyl groups. (Here $\overline{\mathcal{W}} = \overline{\mathcal{W}}(\mathfrak{g}) = \overline{\mathcal{W}}(\widetilde{G})$ is the affine Weyl group corresponding to the simply connected group.) As the Weyl groups act transitively and fixed-point free on the set of simple roots, we have

$$\text{aut } \Phi/\mathcal{W} \cong \Gamma \qquad\qquad \text{aut } \overline{\Phi}/\overline{\mathcal{W}} \cong \overline{\Gamma} , \tag{B.1}$$

where Γ and $\overline{\Gamma}$ are the symmetry groups of the corresponding Dynkin diagrams.
Olive and Turok observed some time ago [26] that

$$\overline{\Gamma}/\mathcal{Z} \cong \Gamma , \tag{B.2}$$

where $\mathcal{Z}$ is the centre of $\widetilde{G}$. In this appendix we want to give a simple geometrical proof of this observation. (A different, algebraic proof has been given in [19].)
Let C_0 be a Weyl chamber of $\mathfrak{g}$ whose walls are made up of r =rank $\mathfrak{g}$ simple roots of $\mathfrak{g}$, and let C_1 be the (unique) alcove of G satisfying $0 \in C_1 \subset C_0$ [16, 4]. (The additional wall of

C_1 corresponds to $(-\psi, 1)$, where ψ is the highest root with respect to the simple roots.) In particular, the origin is a corner of the alcove.
An automorphism of Φ maps C_0 to a Weyl chamber C_0' of $\mathfrak{g}$, and using the transitivity of the action of the Weyl group $\mathcal{W}$ on Weyl chambers, there exists an element in the Weyl group which maps C_0' back to C_0. Furthermore, since $\mathcal{W}$ acts fixed point free, this Weyl group element is uniquely determined. Conversely, every symmetry transformation of C_0 induces a symmetry of the Dynkin diagram, and thus there is a one-to-one correspondence between elements in Γ and symmetry transformations of C_0. We can similarly apply the same argument to the affine Dynkin diagram, and thus conclude that $\overline{\Gamma}$ is in one-to-one correspondence with symmetry transformations of the alcove C_1.
A given symmetry transformation of C_1 corresponds to an element of Γ, if and only if the origin is a fixed point of the symmetry transformation. Furthermore, since all affine hyperplanes intersect at the origin, any symmetry transformation of C_1 must map the origin to a point in $Z \cap C_1$, where Z is the lattice of central elements (which is mapped to the centre of G under exp).
Now, suppose that γ is a symmetry of C_1 which maps the origin to $\gamma(0) \neq 0$. We want to show that there exists an element γ_1 in the semidirect product of the lattice of central elements Z by $\mathcal{W}$,

$$\gamma_1 \in \overline{\mathcal{W}}(\widetilde{G}/\mathcal{Z}) = \mathcal{W} \triangleright Z\,, \tag{B.3}$$

such that γ_1 is a symmetry of C_1 and $\gamma_1 \circ \gamma(0) = 0$. To construct γ_1, we first translate C_1 by $-\gamma(0)$ which is a translation in Z because of the previous argument. Then we use the reflections in the simple roots of $\mathfrak{g}$ to map the translated C_1 back into C_0. It is clear that this defines a symmetry transformation of C_1, and that $\gamma_1 \circ \gamma(0) = 0$.
Conversely, for any point $c \in Z \cap C_1$, we can in this way construct a symmetry transformation of C_1, mapping c to 0. Thus the symmetry transformations of C_1 which do not preserve the affine root are in one-to-one correspondence with the points in $Z \cap C_0$, which in turn is in one-to-one correspondence with the centre of $\widetilde{G}$. As the construction preserves the respective group structures, we have thus shown that (B.2) holds. The proof also implies that

$$\text{aut}\ \overline{\Phi}/\overline{\mathcal{W}}(\widetilde{G}/\mathcal{Z}) \cong \Gamma \cong \text{aut}\ \Phi/\mathcal{W}\,. \tag{B.4}$$

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